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Feynman path integral for dissipative Lagrangians

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Abstract. For a general time-dependent and the Bateman–Morse–Feshbach–Bopp Lagrangian the Feynman path integral is evaluated by establishing a polynomial representation of the Van Vleck determinant.

1. Introduction

Starting with the pioneering work of Feynman (1948), there are many papers and even books dealing with the path integral method. In particular, for quadratic Lagrangians, one can obtain analytical results by using various procedures (Faddeev 1975, Feynman and Hibbs 1965, Gel'fand and Yaglom 1960, Klauder 1960, Marinov 1980, Montroll 1952, Papadopoulos 1978, Schweber 1962). However, there are only a few papers reporting on path integral calculations for the case of a particle which is dissipating energy to the environment with which it is coupled. (On the other hand, one may note that this field has been extensively studied in the framework of other methods; see, for example, Dekker (1981), Messer (1979), Haken (1975), Colegrave and Abdalla (1981, 1982), Vaidyanathan (1982), Remaud and Hernandez (1980), Caldirola and Lugiato (1982), Daniel (1982), Gisin (1981)). A simplified approach could be obtained by considering convenient single particle Lagrangians (Alicki and Messer 1982, Papadopoulos 1974), although a proper quantum mechanical treatment will include the dynamics of both the particle and the environment (Feynman and Vernon 1963, Alicki 1982, Bartnik and Hasse 1982).

This note establishes a single polynomial representation for the Van Vleck determinant or, more precisely, for the result of multiple integration, as defined by Feynman and Hibbs (1965) for quadratic Lagrangians, over the classical trajectories for a fixed 'n' partition of the time interval. Then, the limit $n \to \infty$ appears immediately and the introduction of a convenient differential equation is avoided. We give explicit results for the following two specific Lagrangians:

$$L(q, \dot{q}, t) = \tfrac{1}{2}\mu(t)\dot{q}^2 - \tfrac{1}{2}\mu(t)D(t)\Omega^2 q^2 - a(t)\mu^{1/2}(t)q, \tag{1.1}$$

$$\tilde{L}(q, q^*, \dot{q}, \dot{q}^*, t) = (\dot{q} + \lambda q)(\dot{q}^* - \lambda q^*) - \tilde{\omega}^2 qq^* - a^*(t)q - a(t)q^*, \tag{1.2}$$

where μ, D, a and a^* are functions of time. The Lagrangian (1.1) describes a driven oscillator where mass, frequency and driving force are time dependent. The only

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restrictive conditions we consider are initial conditions:

$$\mu(t_1) = D(t_1) = 1,$$

and the condition

$$D(t)\Omega^2 - \tfrac{1}{4}(2\dot{f}(t) + f^2(t)) = \omega^2 > 0, \tag{1.3}$$

where ω is a constant frequency and the function $f(t)$ is defined as

$$f(t) = \dot{\mu}(t)/\mu(t) \tag{1.4}$$

with $\dot{\mu}(t)$ as the time derivative of $\mu(t)$.

Then, the cases presented by Alicki and Messer (1982) and Papadopoulos (1974) as well as Colegrave and Abdalla (1981, 1982), Landovitz *et al* (1979, 1980), Vaidyanathan (1982), Remaud and Hernandez (1980), in the framework of the Hamiltonian formalism, correspond to a particular choice of the functions μ, D, a and a^*. The Lagrangian (1.2) was introduced first by Bateman (1931) and was rediscovered and analysed subsequently by Morse and Feshbach, Bopp, Feshbach and Tikochinsky (for the corresponding references see the review article of Dekker (1981)). Hereafter we call the Lagrangian (1.2) the BMFB (Bateman, Morse, Feshbach, Bopp) Lagrangian. The corresponding Hamiltonian can be connected with the physical energy by imposing a restriction on the classical trajectories $q(t)$, $q^*(t)$. Then the BMFB Lagrangian leads to the time dependent Caldirola–Kanai Hamiltonian (Dekker 1981, Caldirola and Lugiato 1982). Therefore, the importance of the BMFB Lagrangian lies in the possibility to generate other convenient Lagrangians. However, no path integral calculation with the BMFB Lagrangian has yet been reported in the literature.

2. The calculation of Feynman propagator

According to Feynman (1948) and Feynman and Hibbs (1965), the amplitude of probability (the quantum propagator) to go from q_1 at t_1 to q_2 at t_2 and from (q_1, q_1^*) at t_1 to (q_2, q_2^*) at t_2, respectively, is given for the Lagrangians (1.1) and (1.2) by:

$$K(q_2, t_2; q_1, t_1) = \lim_{n\to\infty} \frac{1}{A_n} \int \exp(\mathrm{i}S_n/\hbar) \prod_{k=1}^{n-1} \frac{\mathrm{d}\xi_k}{A_k}, \tag{2.1}$$

where

$$S_n = \varepsilon \sum_{i=1}^{n} L\left(\xi_{i-1}, \frac{\xi_i - \xi_{i-1}}{\varepsilon}, \tau_{i-1}\right), \qquad n \geqslant 1, \tag{2.2}$$

$$\xi_0 = q_1, \qquad \xi_n = q_2, \qquad \tau_0 = t_1, \qquad \tau_n = t_2,$$

$$\varepsilon = (t_2 - t_1)/n = T/n, \qquad A_k = [2\pi\mathrm{i}\hbar\varepsilon/\mu(\tau_k)]^{1/2}, \qquad k = 1, \ldots, n,$$

and

$$\tilde{K}(q_2, q_2^*, t_2; q_1, q_1^*, t_1) = \lim_{n\to\infty} \frac{1}{AA^*} \int \exp(\mathrm{i}\tilde{S}_n/\hbar) \prod_{k=1}^{n-1} \frac{\mathrm{d}\xi_k}{A} \frac{\mathrm{d}\xi_k^*}{A^*}, \tag{2.3}$$

where

$$\tilde{S}_n = \varepsilon \sum_{i=1}^{n} \tilde{L}\left(\xi_{i-1}, \xi_{i-1}^*, \frac{\xi_i - \xi_{i-1}}{\varepsilon}, \frac{\xi_i^* - \xi_{i-1}^*}{\varepsilon}, \tau_{i-1}\right), \tag{2.4}$$

$$\xi_0 = q_1, \qquad \xi_0^* = q_1^*, \qquad \xi_n = q_2, \qquad \xi_n^* = q_2^*,$$

$$\varepsilon = (t_2 - t_1)/n, \qquad A = [2\pi\mathrm{i}\hbar\varepsilon]^{1/2}, \qquad A^* = [-2\pi\mathrm{i}\hbar\varepsilon]^{1/2}.$$

It is easy to see that the element of integration in (2.1) is defining a path integration over the trajectory $y(t)$ (rather than over $q(t)$), that is given by

$$y(t)=\mu^{1/2}(t)q(t). \tag{2.5}$$

In terms of $y(t)$ the Lagrangian (1.1) becomes

$$L(q,\dot{q},t)=l(y,\dot{y},t)-\tfrac{1}{4}(\mathrm{d}/\mathrm{d}t)(f(t)y^2), \tag{2.6}$$

where

$$l(y,\dot{y},t)=\tfrac{1}{2}\dot{y}^2-\tfrac{1}{2}\omega^2y^2-a(t)y, \tag{2.7}$$

with ω defined by relation (1.3). The Lagrangian $l(y,\dot{y},t)$ describes the usual driven oscillator.

Introducing the classical trajectories one solves the corresponding integrals in (2.1) and (2.3) by using the following formal results of Rzewuski (1969):

$$\int_{-\infty}^{+\infty}\exp(\pm\mathrm{i}x^2)\,\mathrm{d}x=(\pm\mathrm{i}\pi)^{1/2}, \qquad \iint_{-\infty}^{+\infty}\exp(\mathrm{i}xy)\,\mathrm{d}x\,\mathrm{d}y=2\pi. \tag{2.8}$$

The first integral in (2.8) is a sum of two Fresnel integrals

$$\int_{-\infty}^{+\infty}\exp(\pm\mathrm{i}x^2)\,\mathrm{d}x=\sqrt{2\pi}(C(x\to\infty)\pm\mathrm{i}S(x\to\infty))$$

where

$$C(x)=\int_0^x\cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}t, \qquad S(x)=\int_0^x\sin\left(\frac{\pi}{2}t^2\right)\mathrm{d}t.$$

Since $C(x\to\infty)=S(x\to\infty)=\frac{1}{2}$, one finds

$$\int_{-\infty}^{+\infty}\exp(\pm\mathrm{i}x^2)\,\mathrm{d}x=\sqrt{\tfrac{1}{2}\pi}\,(1\pm\mathrm{i})=\sqrt{\pi}\exp(\pm\mathrm{i}\pi/4).$$

The second integral in (2.8) can be formally integrated by using the δ function in terms of a Fourier integral:

$$\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{\infty}\exp(\mathrm{i}xy)\,\mathrm{d}y=\int_{-\infty}^{+\infty}\mathrm{d}x\,2\pi\delta(x)=2\pi.$$

One obtains

$$K(q_2,t_2;q_1,t_1)=\left(\frac{\mu(t_2)}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\lim_{n\to\infty}\left(n\prod_{k=1}^{n-1}f_k(x,1)\right)^{1/2}\exp(\mathrm{i}S_0/\hbar), \tag{2.9}$$

$$\tilde{K}(q_2,q_2^*,t_2;q_1,q_1^*,t_1)=\frac{1}{2\pi\hbar T}\lim_{n\to\infty}\left(n\prod_{k=1}^{n-1}f_k(\tilde{x},\tilde{y})\right)\exp(\mathrm{i}\tilde{S}_0/\hbar), \tag{2.10}$$

where the functions $f_k(\xi,\eta)$, $k=1,2,\ldots$, are defined by a recurrence relation:

$$f_1(\xi,\eta)=1/\xi, \qquad f_{k+1}(\xi,\eta)=(\xi-\eta f_k(\xi,\eta))^{-1}, \tag{2.11}$$

$$x=2-\omega^2T^2/n^2, \qquad \tilde{x}=2-(\tilde{\omega}^2+\lambda^2)T^2/n^2, \qquad \tilde{y}=1-\lambda^2T^2/n^2.$$

Here, S_0, $\tilde{S}_0$ are the values of action along the classical trajectories and are given by

$$S_0 = \frac{\omega}{2\sin\omega T}\left([(y_1^2+y_2^2)\cos\omega T - 2y_1y_2] - 2[y_1Y_2(t_1)+y_2Y_1(t_2)] - 2\int_{t_1}^{t_2} Y_2(s)\dot{Y}_1(s)\,\mathrm{d}s\right) + \tfrac{1}{4}[f(t_1)y_1^2 - f(t_2)y_2^2], \tag{2.12}$$

with

$$Y_i(t) = \frac{1}{\omega}\int_{t_i}^{t} a(s)\sin\omega(s-t_i)\,\mathrm{d}s, \qquad y_i = \mu^{1/2}(t_i)q_i, \qquad i=1,2,$$

and

$$\tilde{S}_0 = \frac{\tilde{\omega}}{\sin\tilde{\omega}T}\left([(\tilde{q}_1\tilde{q}_1^* + \tilde{q}_2\tilde{q}_2^*)\cos\tilde{\omega}T - (\tilde{q}_1^*\tilde{q}_2 + \tilde{q}_1\tilde{q}_2^*)] - [\tilde{q}_1\tilde{Q}_2^*(t_1) + \tilde{q}_2\tilde{Q}_1^*(t_2) + \tilde{q}_1^*\tilde{Q}_2(t_1) + \tilde{q}_2^*\tilde{Q}_1(t_2)] - \int_{t_1}^{t_2}\tilde{Q}_2^*(s)\dot{\tilde{Q}}_1(s)\,\mathrm{d}s - \int_{t_1}^{t_2}\tilde{Q}_2(s)\dot{\tilde{Q}}_1^*(s)\,\mathrm{d}s\right) \tag{2.13}$$

with

$$\tilde{Q}_i(t) = \frac{1}{\tilde{\omega}}\int_{t_i}^{t} a(s)\,\mathrm{e}^{\lambda s}\sin\tilde{\omega}(s-t_i)\,\mathrm{d}s, \qquad \tilde{Q}_i^*(t) = \frac{1}{\tilde{\omega}}\int_{t_i}^{t} a^*(s)\,\mathrm{e}^{-\lambda s}\sin\tilde{\omega}(s-t_i)\,\mathrm{d}s,$$

$$\tilde{q}_i = \mathrm{e}^{\lambda t_i}q_i, \qquad \tilde{q}_i^* = \mathrm{e}^{-\lambda t_i}q_i^*.$$

Furthermore, we have to determine the limits $n\to\infty$ in (2.9) and (2.10). Instead of following the usual procedure (see for example Papadopoulos (1978), Truman (1976) and the references therein) we propose another one:
Let us assume for the functions $f_k(\xi,\eta)$, defined by (2.11), the following representation:

$$f_k(\xi,\eta) = P_{k-1}(\xi,\eta)/P_k(\xi,\eta), \qquad k=1,2,\ldots, \tag{2.14}$$

where $P_k(\xi,\eta)$ are polynomials in two variables. From the recurrence relation (2.11) one gets a corresponding relation for the polynomials $P_k(\xi,\eta)$:

$$P_0(\xi,\eta)=1, \qquad P_1(\xi,\eta)=\xi, \qquad P_{k+1}(\xi,\eta) = \xi P_k(\xi,\eta) - \eta P_{k-1}(\xi,\eta),$$

which yields the results:

$$P_k(\xi,\eta) = \sum_{i=0}^{N_k}(-1)^i\frac{(k-i)!}{(k-2i)!\,i!}\,\xi^{k-2i}\eta^i \tag{2.15}$$

where

$$N_k = k/2 \text{ for } k \text{ even}, \qquad N_k = (k-1)/2 \text{ for } k \text{ odd}.$$

Using (2.14) we obtain for the products appearing in (2.9) and (2.10) the expressions ($n\to n+1$):

$$\lim_{n\to\infty}\left((n+1)\prod_{k=1}^{n} f_k(x,1)\right) = \lim_{n\to\infty}\frac{n+1}{P_n(x,1)}, \qquad \lim_{n\to\infty}\left((n+1)\prod_{k=1}^{n} f_k(\tilde{x},\tilde{y})\right) = \lim_{n\to\infty}\frac{n+1}{P_n(\tilde{x},\tilde{y})}.$$

Since for $n \to \infty$ the variables x, $\tilde{x}$ and $\tilde{y}$ approach the values $x = \tilde{x} = 2$ and $\tilde{y} = 1$, we expand $P_n(x, 1)$ and $P_n(\tilde{x}, \tilde{y})$ in Taylor series about $x = \tilde{x} = 2$ and $\tilde{y} = 1$. For that purpose we have to introduce the derivatives of $P_k(\xi, \eta)$ at $\xi = 2$ and $\eta = 1$, defined by

$$P_k^{s-u,u} \equiv \frac{\partial^{s-u}}{\partial \xi^{s-u}} \frac{\partial^u}{\partial \eta^u} P_k(\xi, \eta) \bigg|_{\substack{\xi=2 \\ \eta=1}}.$$

Inserting the polynomials (2.15) we calculate

$$P_k^{s-u,u} = (-1)^u \frac{s!}{(2s+1)!} \frac{(k+1+s-u)!}{(k-s-u)!}. \tag{2.16}$$

In particular, we have:

$$\lim_{n\to\infty} \frac{1}{n^{2s+1}} P_n^{s-u,u} = (-1)^u \frac{s!}{(2s+1)!} \qquad \text{for } s \text{ and } u \text{ fixed,} \tag{2.17}$$

and thus, by using a Taylor expansion, we get the result

$$\lim_{n\to\infty} \frac{1}{n+1} P_n\left(2 - \frac{(\omega^2+\lambda^2)T^2}{n^2}, 1 - \frac{\lambda^2 T^2}{n^2}\right) = \frac{\sin \omega T}{\omega T}. \tag{2.18}$$

Now the propagators (2.9) and (2.10) are completely determined. If we choose $D(t) = \mu(t) = 1$ in the Lagrangian (1.1), then we recognise in (2.9) the well known Green function for a driven oscillator with unit mass. In the limit $\Omega \to 0$ and $\mu(t) = 1$ we obtain the case of a particle with unit mass driven by a force. If we choose $D(t) = 1$ and $\mu(t) = e^{-\lambda(t-t_1)}$, we get the results presented by Papadopoulos (1974), Colegrave and Abdalla (1981) and Landovitz *et al* (1979).

If we impose for the propagator (2.10) the conditions

$$q_1 e^{\lambda t_1} = q_1^* e^{-\lambda t_1}, \qquad q_2 e^{\lambda t_2} = q_2^* e^{-\lambda t_2}, \qquad a(t) = a^*(t) = 0, \tag{2.19}$$

then the classical action (2.13) becomes:

$$\tilde{S}_0 = \frac{\tilde{\omega}}{2 \sin \tilde{\omega} T} [(\tilde{q}_1^2 + \tilde{q}_2^2) \cos \tilde{\omega} T - 2\tilde{q}_1\tilde{q}_2] + \frac{\tilde{\omega}}{2 \sin \tilde{\omega} T} [(\tilde{q}_1^{*2} + \tilde{q}_2^{*2}) \cos \tilde{\omega} T - 2\tilde{q}_1^*\tilde{q}_2^*]. \tag{2.20}$$

This result means that under the conditions (2.19) the propagator of the BMFB Lagrangian is the product of the propagators of two time-dependent oscillators of the type (1.1). The masses of these oscillators are varying according to $\mu(t) \sim e^{-2\lambda t}$ and $e^{2\lambda t}$, but their frequencies remain the same $\tilde{\omega}$. This result corresponds to the transformation of the BMFB Hamiltonian into a sum of two Caldirola–Kanai Hamiltonians by using a sequence of canonical transformations (Dekker 1981).

3. Concluding remarks

We have calculated the Feynman path integral for two Lagrangians, which could account for the dissipation effects. This was done by establishing a single polynomial representation for the Van Vleck determinants.

Following the ideas of Feynman and Hibbs (1965), we could suppose that the path integral gives the corresponding quantum propagators, provided the usual semigroup properties of time evolution are satisfied. In that case we can build up the whole quantum mechanics, namely the Schrödinger equation, time evolution operators, and the other quantities. It would then be interesting to compare the quantum mechanics obtained in this manner with that from other methods (see the reviews Dekker (1981), Messer (1979), and also the references Daniel (1982), Gisin (1981)). This work is in progress.

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